# MA40050: Numerical Optimisation & Large–Scale Systems

## Review of Eigenvalues and Eigenvectors

For any symmetric $A \in \mathbb{R}^{N\times N}$ there exist eigenvalues $\lambda_1 \le \cdots \le \lambda_N \in \mathbb{R}$ and eigenvectors $v_1, \ldots v_N \in \mathbb{R}^N$ such that

$$Av_n = \lambda_n v_n, \qquad n = 1, \ldots, N.$$

The set $\{v_1, \ldots, v_N\}$ is an orthonormal basis of $\mathbb{R}^N$. We call $\sigma(A) := \{\lambda_1, \ldots, \lambda_N\}$ the *spectrum* of $A$.

Moreover, we have the following properties:

1. $A$ has the spectral decomposition $A = QDQ^T$ where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ and $Q = (v_1|\ldots|v_N)$. The matrix $Q$ is orthogonal, i.e., $Q^{-1} = Q^T$ and $|Qx| = |x|$ for all $x$. This representation (into an orthogonal and a diagonal matrix) is unique up to a permutation of the columns of $D$ and $Q$.

   **Proof:** $QDQ^T v_n = \lambda_n v_n = Av_n$, $\forall n$. Since $\{v_n\}_{n=1}^N$ is a basis, result follows by linearity.

2. $A$ is invertible if, and only if, $0 \notin \sigma(A)$

   **Proof:** $0 \in \sigma(A) \quad \Leftrightarrow \quad \exists v \in \mathbb{R}^N \setminus \{0\}$ s.t. $Av = 0v = 0 \quad \Leftrightarrow \quad A$ is not 1-1.

3. If $A$ is invertible then $\sigma(A^{-1}) = \{1/\lambda_1, \ldots, 1/\lambda_N\}$, and the eigenbasis is the same.

   **Proof:** $Av_n = \lambda_n v_n \quad \Rightarrow \quad v_n = \lambda_n A^{-1} v_n \quad \Rightarrow \quad A^{-1} v_n = \frac{1}{\lambda_n} v_n$, since $\lambda_n \neq 0 \; \forall \, n$.

4. $\|A\| = \max_{n=1,\ldots N} |\lambda_n|$ and $\|A^{-1}\| = 1/\min|\lambda_n|$. In particular, $\kappa(A) = \max|\lambda_n|/\min|\lambda_n|$.

   **Proof:** $|Ax| = |QDQ^T x| = |D(Q^T x)| \le \|D\||Q^T x| = \max_{n\le N} |\lambda_n||x|$, $\forall\, x \in \mathbb{R}^N$. Equality is attained for a suitable eigenvector. A similar argument shows the result for $\|A^{-1}\|$.

5. $x^T Ax \ge \min_n \lambda_n |x|^2$, $\; x \in \mathbb{R}^N$. In particular, $A$ is spd if, and only if, $\lambda_n > 0$ for all $n$.

   **Proof:** $x^T Ax = x^T QDQ^T x = (Q^T x)D(Q^T x) \ge \min \lambda_n |Q^T x|^2 = \min \lambda_n |x|^2$. Equality is attained if $x$ is an appropriate eigenvector.

6. $A$ is spd if, and only if, $A^{-1}$ is spd.

   **Proof:** Obvious from 1.

7. If $A$ is positive semi-definite, then $\sqrt{A} := A^{1/2} := Q\operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N})Q^T$ is symmetric and positive semidefinite, and satisfies $(A^{1/2})^2 = A$. (In fact, it is the unique symmetric and positive semidefinite matrix which satisfies this.) If $A$ Is spd then $A^{1/2}$ is spd.

   **Proof:** The first part is obvious, the uniqueness is a little more difficult (but we won't need it in this course.)